

CONDITIONAL ESTIMATES ON SMALL DISTANCES BETWEEN ORDINATES OF ZEROS OF $\zeta(s)$ AND $\zeta'(s)$

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ABSTRACT. Let $\beta' + i\gamma'$ be a zero of $\zeta'(s)$. In [3] Garaev and Yıldırım proved that there is a zero $\beta + i\gamma$ of $\zeta(s)$ with $\gamma' - \gamma \ll \sqrt{|\beta' - 1/2|}$. Assuming RH, we improve this bound by saving a factor $\sqrt{\log \log \gamma'}$.

1. INTRODUCTION

The distribution of zeros of the Riemann zeta-function $\zeta(s)$ is closely connected to that of zeros of $\zeta'(s)$. As just one illustration we cite A. Speiser's [6] theorem that the Riemann Hypothesis (RH) is equivalent to the nonexistence of non-real zeros of $\zeta'(s)$ in the half-plane $\Re s < 1/2$.

Let $\rho' = \beta' + i\gamma'$ be a zero of $\zeta'(s)$, and let $\rho_c = \rho_c(\rho') = \beta_c + i\gamma_c$ be a zero of $\zeta(s)$ with smallest $|\gamma' - \gamma_c|$ (if there is more than one such zero, take any of them). M. Z. Garaev and C. Y. Yıldırım [3] showed that

$$\gamma' - \gamma_c \ll \sqrt{|\beta' - 1/2|}.$$

Their result is unconditional. Our purpose here is to obtain a conditional improvement.

Theorem 1. *Assume RH. We have*

$$\gamma' - \gamma_c \ll \sqrt{\frac{\beta' - 1/2}{\log \log \gamma'}} \tag{1}$$

for $\beta' - 1/2 \leq 1/\log \log \gamma'$. Here the implied constant is absolute, and for γ' sufficiently large we may take the implied constant to be 2.16.

Remark 1. Note that on RH we trivially have

$$\gamma' - \gamma_c \ll \frac{1}{\log \log \gamma'}. \tag{2}$$

Combining this with our Theorem 1, we see that on RH

$$\gamma' - \gamma_c \ll \min \left\{ \sqrt{\frac{\beta' - 1/2}{\log \log \gamma'}}, \frac{1}{\log \log \gamma'} \right\}.$$

The inequality (2) follows from the well-known fact that on RH, the largest gap between consecutive zeros of $\zeta(s)$ up to height T is $\ll 1/\log \log T$ (see [8], for example).

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Remark 2. In [2] D. W. Farmer, S. M. Gonek and C. P. Hughes conjectured that

$$\limsup_{t \rightarrow \infty} \frac{S(t)}{\sqrt{\log t \log \log t}} = \frac{1}{\pi\sqrt{2}}.$$

Assuming this as well as RH, one can show (by the same proof as that of Theorem 1) that

$$\gamma' - \gamma_c \ll \sqrt{\beta' - 1/2} \left(\frac{\log \log \gamma'}{\log \gamma'} \right)^{1/4}$$

for $\beta' - 1/2 \ll \sqrt{\log \log \gamma' / \log \gamma'}$.

Remark 3. There are multiple ways to prove results like Theorem 1. For example, one can start with Lemma 2 below, split the sum into three parts (according to $|\gamma - \gamma'| \leq 1/\log \log \gamma'$, $1/\log \log \gamma' < |\gamma - \gamma'| \leq 1$ or $|\gamma - \gamma'| \geq 1$), and estimate each part separately. This will give a slightly weaker result than Theorem 1. The proof we present in this paper follows another clue, which we think is more inspiring and more likely to be modified. For example, with a little more care it is possible to show that (on RH) for $\beta' - 1/2 \leq 1/\log \log \gamma'$ there are $\gg (\beta' - 1/2) \log \gamma'$ zero(s) of $\zeta(s)$ lie in $\left[\gamma' - C\sqrt{\frac{\beta' - 1/2}{\log \log \gamma'}}, \gamma' + C\sqrt{\frac{\beta' - 1/2}{\log \log \gamma'}} \right]$ for some constant C .

2. LEMMAS

Lemma 2. Assume RH. If $\beta' > 1/2$, then we have

$$\frac{\log \gamma'}{2} = \sum_{\gamma} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - \gamma)^2} + O(1).$$

See equation (4) in [7].

Let $N(T) = \sum_{0 < \gamma \leq T} 1$ be the zero counting function of $\zeta(s)$. It is well-known (see [8]) that

$$N(T) = L(T) + S(T) + E(T),$$

where

$$L(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \frac{7}{8}, \quad S(T) = \pi^{-1} \arg \zeta(1/2 + iT),$$

and $E(T)$ is an error term. We require the following result.

Lemma 3. We have

$$d(L(u) + E(u)) = \left(\frac{1}{2\pi} \log u + O(1) \right) du.$$

Proof. By the proof of Theorem 9.3 in [8] we know that

$$L(T) + E(T) = 1 - \frac{T \log \pi}{2\pi} + \frac{1}{\pi} \Im \log \Gamma(1/4 + iT/2).$$

Therefore, we have

$$d(L(u) + E(u)) = \left(\frac{1}{\pi} \frac{d \Im \log \Gamma(1/4 + iu/2)}{du} + O(1) \right) du.$$

It is straightforward to compute that

$$\frac{d \Im \log \Gamma(1/4 + iu/2)}{du} = \frac{1}{2} \Re \frac{\Gamma'}{\Gamma}(1/4 + iu/2).$$

By Stirling's formula, this is $(\log u)/2 + O(1)$. Hence the result. \square

Lemma 4. *Let $T > 2$ and $T < t_1 < t_2 < 2T$. Then we have*

$$E(t_2) - E(t_1) \ll t_2 - t_1.$$

Proof. Write $s = 1/2 + it$. By the proof of Theorem 9.3 in [8] we know that

$$\begin{aligned} \pi E(t) &= \pi(N(t) - S(t) - L(t)) \\ &= \Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg \zeta(s) \\ &\quad - \arg \zeta(s) - \frac{T}{2} \log t + \frac{1 + \log 2\pi}{2} t - \frac{7}{8} \pi \\ &= \Delta \arg \Gamma(s/2) - \frac{T}{2} \log t + \frac{1 + \log 2}{2} t + \frac{\pi}{8}. \end{aligned}$$

It follows that

$$\pi(E(t_2) - E(t_1)) = \Delta \arg \Gamma(1/4 + it_2/2) - \Delta \arg \Gamma(1/4 + it_1/2) - \frac{1}{2}(t_2 - t_1) \log T + O(t_2 - t_1).$$

By the mean value theorem of calculus,

$$\Delta \arg \Gamma(1/4 + it_2/2) - \Delta \arg \Gamma(1/4 + it_1/2) = (t_2 - t_1) \cdot \frac{1}{2} \Re \frac{\Gamma'}{\Gamma}(1/4 + it_3/2)$$

for some $t_3 \in [t_1, t_2]$. But this is

$$\frac{1}{2}(t_2 - t_1) \log T + O\left(\frac{t_2 - t_1}{T}\right)$$

by Stirling's formula. Hence the result. \square

3. PROOF OF THE THEOREM

It is well-known that

$$\zeta'(1/2 + i\gamma') = 0 \implies \zeta(1/2 + i\gamma') = 0.$$

Therefore, $\beta' = 1/2$ implies that $\gamma_c = \gamma'$, in which case (1) is trivially true. Below we assume that $1/2 < \beta' \leq 1/2 + 1/\log \log \gamma'$. We may also assume $\gamma' > 2015$ for convenience.

Define

$$h(t) = h_{\rho'}(t) = \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (t - \gamma')^2}.$$

By Lemma 2 we have

$$\sum_{\gamma} h(\gamma) = \frac{1}{2} \log \gamma' + O(1).$$

It is well-known that $\zeta(s)$ has no zero in the region

$$\sigma > 0, \quad -14 \leq t \leq 14.$$

For $t \leq -14$, there are $\ll \log |t|$ zeros $1/2 + i\gamma$ of $\zeta(s)$ for which $t - 1 \leq \gamma \leq t$. Thus, it is easy to see that

$$\sum_{-\infty < \gamma \leq 14} h(\gamma) = \sum_{n=14}^{\infty} \sum_{-n-1 < \gamma \leq -n} h(\gamma) \ll (\beta' - 1/2) \cdot \sum_{n=1}^{\infty} \frac{\log n}{n^2} \ll 1.$$

It follows that

$$\frac{1}{2} \log \gamma' + O(1) = \sum_{\gamma > 14} h(\gamma) = \int_{14}^{\infty} h(u) d(N(u)) = \int_{14}^{\infty} h(u) d(L(u) + E(u) + S(u)). \quad (3)$$

Next we show that

$$\int_{14}^{\infty} h(u) d(L(u) + E(u)) = \frac{\log \gamma'}{2} + O(1).$$

By Lemma 3 we have

$$\int_{14}^{\infty} h(u) d(L(u) + E(u)) = \int_{14}^{\infty} h(u) \left(\frac{\log u}{2\pi} + O(1) \right) du.$$

It is clear that

$$\left(\int_{14}^{\gamma'/2} + \int_{3\gamma'/2}^{\infty} \right) \left(h(u) \left(\frac{\log u}{2\pi} + O(1) \right) \right) du \ll 1,$$

and that

$$\int_{\gamma'/2}^{3\gamma'/2} h(u) \left(\frac{\log u}{2\pi} + O(1) \right) du = \frac{\log \gamma'}{2\pi} \int_{\gamma'/2}^{3\gamma'/2} h(u) du + O(1).$$

Hence, we see that

$$\int_{14}^{\infty} h(u) d(L(u) + E(u)) = \frac{\log \gamma'}{2\pi} \int_{\gamma'/2}^{3\gamma'/2} h(u) du + O(1).$$

Now we plainly have

$$\int_{\gamma'/2}^{3\gamma'/2} h(u) du = 2 \arctan \left(\frac{\gamma'}{2(\beta' - 1/2)} \right) = \pi + O \left(\frac{\beta' - 1/2}{\gamma'} \right).$$

Therefore, we obtain

$$\int_{14}^{\infty} h(u) d(L(u) + E(u)) = \frac{\log \gamma'}{2} + O(1).$$

This together with (3) give us

$$\int_{14}^{\infty} h(u) dS(u) = O(1).$$

By integration by parts, we see that

$$\int_{14}^{\infty} h(u) dS(u) = - \int_{14}^{\infty} h'(u) S(u) du + O(1).$$

It follows that

$$\int_{14}^{\infty} h'(u)S(u)du = O(1). \quad (4)$$

Let $p = p(\gamma')$ be a parameter to be determined later. Split the above integral into three parts:

$$\int_{14}^{\infty} h'(u)S(u)du = \left[\left(\int_{14}^{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}}^{2\gamma'} \right) + \int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{2\gamma'}^{\infty} \right] h'(u)S(u)du.$$

We estimate them separately. First, since

$$h'(u) = -\frac{2(u - \gamma')(\beta' - 1/2)}{((\beta' - 1/2)^2 + (u - \gamma')^2)^2},$$

we trivially have

$$\int_{2\gamma'}^{\infty} h'(u)S(u)du \ll 1/\gamma'. \quad (5)$$

Next we consider

$$\left(\int_{14}^{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}}^{2\gamma'} \right) h'(u)S(u)du.$$

It is straightforward to compute that

$$\int_{-\infty}^{\infty} |h'(u)|du = 2 \int_{-\infty}^{\gamma'} h'(u)du = 2h(u) \Big|_{-\infty}^{\gamma'} = \frac{2}{\beta' - 1/2},$$

and that

$$\int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} |h'(u)|du = \frac{2}{\beta' - 1/2} \cdot \frac{1}{1 + (\beta' - 1/2)p^2}.$$

Hence, using the bound (see [8])

$$|S(T)| \leq \frac{A \log T}{\log \log T}$$

for some absolute positive constant A , we see that

$$\begin{aligned} & \left(\int_{14}^{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}}^{2\gamma'} \right) |h'(u)S(u)|du \\ & \leq \frac{2A \log \gamma'}{\log \log \gamma'} \left(\int_{14}^{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}}^{2\gamma'} \right) |h'(u)|du \\ & \leq \frac{2A \log \gamma'}{\log \log \gamma'} \left(\int_{-\infty}^{\infty} - \int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} \right) |h'(u)|du \\ & = \frac{2A \log \gamma'}{\log \log \gamma'} \left(\frac{2}{\beta' - 1/2} - \frac{2}{\beta' - 1/2} \cdot \frac{1}{1 + (\beta' - 1/2)p^2} \right) \\ & = \frac{2A \log \gamma'}{\log \log \gamma'} \cdot \frac{2p^2}{1 + (\beta' - 1/2)p^2}. \end{aligned} \quad (6)$$

Now we turn to

$$\int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} h'(u) S(u) du .$$

Suppose that there is no zero of $\zeta(s)$ on the vertical segment

$$\left[1/2 + i\left(\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}\right), 1/2 + i\left(\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}\right) \right] . \quad (7)$$

Then we have $N(t_2) - N(t_1) = 0$ for $t_1, t_2 \in \left[\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}, \gamma' + \frac{\sqrt{\beta' - 1/2}}{p} \right]$. It follows that

$$S(t_1) - S(t_2) = L(t_2) - L(t_1) + E(t_2) - E(t_1) = \frac{t_2 - t_1}{2\pi} \log \gamma' + O(t_2 - t_1) + E(t_2) - E(t_1).$$

By Lemma 4, this is

$$\frac{t_2 - t_1}{2\pi} \log \gamma' + O(t_2 - t_1). \quad (8)$$

Therefore, since

$$h'(u) = -\frac{2(u - \gamma')(\beta' - 1/2)}{((\beta' - 1/2)^2 + (u - \gamma')^2)^2},$$

by changing variables we see that

$$\int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} h'(u) S(u) du = \int_0^{\frac{\sqrt{\beta' - 1/2}}{p}} \frac{2(\beta' - 1/2)v}{((\beta' - 1/2)^2 + v^2)^2} \left(S(\gamma' - v) - S(\gamma' + v) \right) dv .$$

By (8), this is

$$\int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} h'(u) S(u) du = \int_0^{\frac{\sqrt{\beta' - 1/2}}{p}} \frac{4(\beta' - 1/2)v^2}{((\beta' - 1/2)^2 + v^2)^2} \left(\frac{\log \gamma'}{2\pi} + O(1) \right) dv ,$$

and a straightforward computation turns it into

$$\int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} h'(u) S(u) du = \left(\frac{\log \gamma'}{\pi} + O(1) \right) \cdot \left(\frac{-p\sqrt{\beta' - 1/2}}{1 + (\beta' - 1/2)p^2} + \arctan \left(\frac{1}{p\sqrt{\beta' - 1/2}} \right) \right).$$

Combining this with (4), (5) and (6) we obtain

$$\begin{aligned} & \left(\frac{\log \gamma'}{\pi} + O(1) \right) \cdot \left(\frac{-p\sqrt{\beta' - 1/2}}{1 + (\beta' - 1/2)p^2} + \arctan \left(\frac{1}{p\sqrt{\beta' - 1/2}} \right) \right) \\ &= \int_{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}}^{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}} h'(u) S(u) du \\ &= \left[\int_{14}^{\infty} - \left(\int_{14}^{\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}} + \int_{\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}}^{2\gamma'} \right) - \int_{2\gamma'}^{\infty} \right] h'(u) S(u) du \\ &\leq O(1) + \frac{2A \log \gamma'}{\log \log \gamma'} \cdot \frac{2p^2}{1 + (\beta' - 1/2)p^2} + O(1/\gamma'). \end{aligned} \quad (9)$$

We wish to choose $p = c\sqrt{\log \log \gamma'}$ for some sufficiently small positive constant c such that

$$\frac{-p\sqrt{\beta' - 1/2}}{1 + (\beta' - 1/2)p^2} + \arctan \left(\frac{1}{p\sqrt{\beta' - 1/2}} \right) \geq \frac{\pi}{3}, \quad (10)$$

and that

$$\frac{2A}{\log \log \gamma'} \cdot \frac{2p^2}{1 + (\beta' - 1/2)p^2} \leq \frac{1}{6}. \quad (11)$$

We show such c exists. In fact, we clearly have

$$\frac{2A}{\log \log \gamma'} \cdot \frac{2p^2}{1 + (\beta' - 1/2)p^2} = \frac{4Ac^2}{1 + c^2(\beta' - 1/2) \log \log \gamma'} \leq 4Ac^2.$$

Next, since $\beta' - 1/2 \leq 1/\log \log \gamma'$ we have $0 < p\sqrt{\beta' - 1/2} \leq c$. It follows that

$$\frac{-p\sqrt{\beta' - 1/2}}{1 + (\beta' - 1/2)p^2} + \arctan\left(\frac{1}{p\sqrt{\beta' - 1/2}}\right) \geq -c + \arctan(c^{-1}).$$

Thus, there does exist a small constant $c > 0$ such that both (10) and (11) hold.

Now combining (9) with (10) and (11), we obtain

$$\left(\frac{\log \gamma'}{\pi} + O(1)\right) \cdot \frac{\pi}{3} \leq \frac{\log \gamma'}{6} + O(1),$$

which is clearly a contradiction for large γ' .

Hence, the assumption (7) must be false. This means there exists a zero of $\zeta(s)$ on the vertical segment

$$\left[1/2 + i\left(\gamma' - \frac{\sqrt{\beta' - 1/2}}{p}\right), 1/2 + i\left(\gamma' + \frac{\sqrt{\beta' - 1/2}}{p}\right)\right].$$

This ends our proof. □

Note added in proof: From the above discussion we see that for any $\epsilon > 0$ and γ' sufficiently large (depending on ϵ), it suffices to choose c such that

$$-c + \arctan(c^{-1}) \geq \pi \cdot 4Ac^2 + \epsilon.$$

By the work of E. Carneiro, V. Chandee and M. B. Milinovich [1], we can take $A = \frac{1}{4} + o(1)$. Therefore, we may choose any positive $c < c_0$ where c_0 is the positive root of $\arctan(x^{-1}) - x = \pi x^2$, whose numerical value is $c_0 = 0.463\dots$. Thus, for γ' sufficiently large, we may then take the implied constant in (1) to be $1/0.463 \approx 2.16$.

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